June 2006 C4

1) $8x + 2x\frac{dy}{dx} + 2y + 2y\frac{dy}{dx} = 0$ $\quad \frac{dy}{dx} = \frac{-8x-2y}{2x+2y} = \frac{-(4x+y)}{x+y}$.

at $(1,2)$ $\frac{dy}{dx} = \frac{-6}{3} = -2$.

2) i $(1-3x)^{-2} = 1 + -2(-3x) + \frac{(-2)(-3)}{2}(-3x)^2 + \quad = 1 + 6x + 27x^2 + \ldots$

ii $(1+2x)^2 = 1 + 4x + 4x^2$

$(1+2x)^2(1-3x)^{-2} = (1+4x+4x^2)(1+6x+27x^2+\ldots$

coefficient x^2 $\quad 27x^2 + 24x^2 + 4x^2 \quad 55x^2$ $\quad$ coefficient x^2 is 55.

3) $\frac{3-2x}{x(3-x)} \equiv \frac{A}{x} + \frac{B}{3-x}$ $\quad 3-2x \equiv A(3-x) + Bx$ $\quad x=0 \quad 3 = 3A \quad A=1$

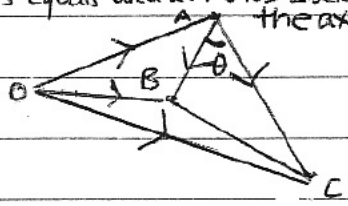
$x=3 \quad -3 = 3B \quad B=-1$

$\frac{3-2x}{x(3-x)} = \frac{1}{x} - \frac{1}{3-x}$

$\int_1^2 \frac{3-2x}{x(3-x)}dx = \int_1^2 \frac{1}{x} - \frac{1}{3-x}\,dx = \left[\ln x + \ln|3-x|\right]_1^2 = (\ln 2 + \ln 1) - (\ln 1 + \ln 2) = 0$

the area $x=1$ to $x=1.5$ above x axis equals area $x=1.5$ to $x=2$ below the axis.

4 $\vec{AB} = \underline{b} - \underline{a} = \begin{pmatrix}4\\2\\-4\end{pmatrix} - \begin{pmatrix}7\\3\\-3\end{pmatrix} = \begin{pmatrix}-3\\-1\\-1\end{pmatrix}$ $\quad \vec{AC} = \underline{c} - \underline{a} = \begin{pmatrix}5\\4\\-5\end{pmatrix} - \begin{pmatrix}7\\3\\-3\end{pmatrix} = \begin{pmatrix}-2\\1\\-2\end{pmatrix}$



(i) $AB \cdot AC = |AB||AC|\cos\theta$ $\quad \cos\theta = \frac{\begin{pmatrix}-3\\-1\\-1\end{pmatrix} \cdot \begin{pmatrix}-2\\1\\-2\end{pmatrix}}{\sqrt{9+1+1}\,\sqrt{4+1+4}}$

$\cos\theta = \frac{6 + -1 + 2}{\sqrt{11}\sqrt{9}} = \frac{7}{3\sqrt{11}}$ $\quad \underline{\theta = 45.3^\circ \text{ (3sf)}}$

(ii) $\cos\theta = \frac{7}{3\sqrt{11}}$ $\quad \cos^2\theta = \frac{49}{9 \times 11}$ $\quad \sin^2\theta = 1 - \cos^2\theta = 1 - \frac{49}{99} = \frac{50}{99}$ $\quad \sin\theta = \frac{5\sqrt{2}}{3\sqrt{11}}$

Area $= \frac{1}{2}\sqrt{11} \times 3 \sin\theta = \frac{\cancel{3}}{2}\frac{\sqrt{11} \times 5\sqrt{2}}{\cancel{3}\sqrt{11}} = \frac{5\sqrt{2}}{2}$.

5 $\frac{dA}{dt} = kA^2$ $\quad \int \frac{1}{A^2}dA = \int k\,dt$ $\quad -\frac{1}{A} = kt + c$ $\quad -\frac{1}{1000} = k + c$ $\quad -\frac{1}{2000} = 2k + c$

$-\frac{1}{2000} - \left(-\frac{1}{1000}\right) = k$ $\quad k = \frac{1}{2000}$ $\quad -\frac{1}{1000} = \frac{1}{2000} + c$ $\quad c = -\frac{3}{2000}$

$-\frac{1}{A} = \frac{1}{2000}t - \frac{3}{2000}$ $\quad A = 3000$ $\quad -\frac{1}{3000} = \frac{1}{2000}(t-3)$ $\quad -\frac{2}{3} = t - 3$

$t = 2\frac{1}{3}$ hours

6. $u = e^x + 1$ $\quad \frac{du}{dx} = e^x$ $\quad \frac{dx}{du} = e^{-x}$ $\quad e^x = u - 1$ $\quad \frac{dx}{du} = \frac{1}{u-1}$

$\int \frac{e^{2x}}{e^x+1}dx = \int \frac{(e^x)^2}{e^x+1}dx = \int \frac{(u-1)^2}{u} \times \frac{1}{(u-1)}du = \int \frac{(u-1)}{u}du$

$x=1 \quad u = e^1 + 1 \quad x = 0 \quad u = 2$.

$$\int_0^1 \frac{e^{2x}}{e^{x}+1}\,dx = \int_2^{e^1+1} \frac{u-1}{u}\,du = \int_2^{e^1+1} 1-\frac{1}{u}\,du = \Big[u - \ln|u|\Big]_2^{e^1+1}$$

$$(e^1+1 - \ln|e^1+1|) - (2 - \ln|2|) = e^1-1-(\ln|e^1+1| - \ln|2|) = e^1-1-\ln\left(\frac{e+1}{2}\right)$$

7. $r = \begin{pmatrix}1\\-2\\4\end{pmatrix} + \lambda\begin{pmatrix}3\\1\\a\end{pmatrix}$ $\quad r = \begin{pmatrix}-8\\2\\3\end{pmatrix} + \mu\begin{pmatrix}1\\-2\\-1\end{pmatrix}$ lines are not parallel.

Skew lines do not intersect.
$$1+3\lambda = -8+\mu \quad (x)$$
$$-2+\lambda = 2-2\mu \quad (y)$$
$$4+a\lambda = 3-\mu \quad (z).$$

Use x and y to find λ and μ. $\quad 9 = \mu - 3\lambda$, $-4 = -2\mu - \lambda$ $\quad 18 = 2\mu - 6\lambda$, $-4 = -2\mu - \lambda$ $\quad 14 = -7\lambda$ $\lambda = -2$, $\mu = 3$

(i) for lines to intersect $\quad 4 + -2a = 3-3 \quad a = 2 \quad$ lines are skew when $a \neq 2$.

(ii) $a=2$. $\quad \begin{pmatrix}1-6\\-2-2\\4-4\end{pmatrix} = \begin{pmatrix}-5\\-4\\0\end{pmatrix}$ $\quad$ check. $\begin{pmatrix}-8+3\\2-6\\3-3\end{pmatrix} = \begin{pmatrix}-5\\-4\\0\end{pmatrix}$ is point of intersection.

8. $\cos 2x = 2\cos^2 x - 1 \quad \cos^2 x = \frac{1}{2}(\cos 2x + 1) \quad \int \cos^2 6x\,dx = \int \frac{1}{2}(\cos 12x + 1)\,dx.$

$$\frac{1}{2}\int \cos 12x + 1\,dx = \frac{1}{2}\left(\frac{\sin 12x}{12} + x\right) + c = \frac{1}{2}x + \frac{\sin 12x}{24} + c$$

$$\int_0^{\frac{\pi}{12}} \underset{u}{x}\ \underset{\frac{dv}{dx}}{\cos^2 6x}\,dx = x\left(\frac{x}{2} + \frac{\sin 12x}{24}\right) - \int \frac{x}{2} + \frac{\sin 12x}{24}\,dx$$

$u = x \quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = \cos^2 6x \quad v = \frac{x}{2} + \frac{\sin 12x}{24}$

$$= \left[x\left(\frac{x}{2} + \frac{\sin 12x}{24}\right) - \left(\frac{x^2}{4} + -\frac{\cos 12x}{12\times 24}\right)\right]_0^{\pi/12} = \left(\frac{\pi}{12}\left(\frac{\pi}{24}+0\right) - \left(\frac{\pi^2}{4\times144} + \frac{1}{12\times24}\right)\right) - \left(0 - \left(0 - \frac{1}{12\times24}\right)\right)$$

$$= \frac{\pi^2}{12\times24} - \frac{\pi^2}{4\times144} - \frac{1}{12\times24} - \frac{1}{12\times24} = \frac{1}{576}(\pi^2 - 4).$$

9. $x = 4\cos t \quad y = 3\sin t \quad \frac{dx}{dt} = -4\sin t \quad \frac{dy}{dt} = 3\cos t \quad \frac{dy}{dx} = \frac{3\cos t}{-4\sin t}$

$\frac{dy}{dx} = -\frac{3}{4}\cot t \quad t = p. \quad y - 3\sin p = \frac{-3\cos p}{4\sin p}(x - 4\cos p)$

$4y\sin p - 12\sin^2 p = -3x\cos p + 12\cos^2 p \quad 4y\sin p + 3x\cos p = 12$

at R $y=0$ $x = \frac{4}{\cos p}$ $\quad$ at S $x=0$ $y = \frac{3}{\sin p}$ $\quad$ area of triangle $\frac{1}{2}xy$

$$\text{area} = \frac{1}{2}\,\frac{4}{\cos p}\,\frac{3}{\sin p} = \frac{12}{\sin 2p}.$$

least possible value of area of triangle is 12 when $\sin 2p = 1$ $\quad 2p = \frac{\pi}{2}$
when $p = \frac{\pi}{4}$